Appendix 3.3

Contents

1. **Example 3.3.16** *With* $f(x) = e^x(\sin x + \cos x)$ *calculate* $T_{8,0}f(x)$.

   Solution

$$
\begin{aligned}
f^{(1)}(x) &= e^x(\sin x + \cos x) + e^x(\cos x - \sin x) \\
&= 2e^x \cos x, \\
f^{(2)}(x) &= 2e^x \cos x - 2e^x \sin x \\
&= 2e^x(\cos x - \sin x), \\
f^{(3)}(x) &= 2e^x(\cos x - \sin x) + 2e^x(-\sin x - \cos x) \\
&= -4e^x \sin x. \\
f^{(4)}(x) &= -4e^x \sin x - 4e^x \cos x = -4f(x).
\end{aligned}
$$

   The fact that a derivative is connected to the function simplifies matters greatly. For now

$$
\begin{aligned}
f^{(5)}(x) &= -4f^{(1)}(x), \; f^{(6)}(x) = -4f^{(2)}(x), \; f^{(7)}(x) = -4f^{(3)}(x) \\
\text{and} \quad f^{(8)}(x) &= -4f^{(4)}(x) = 16f(x).
\end{aligned}
$$

   Thus $f(0) = 1$, $f^{(1)}(0) = 2$, $f^{(2)}(0) = 2$, $f^{(3)}(0) = 0$, $f^{(4)}(0) = -4$, $f^{(5)}(0) = -8$, $f^{(6)}(0) = -8$, $f^{(7)}(0) = 0$ and $f^{(8)}(0) = 16$.

   Hence

$$
\begin{aligned}
T_{8,0}f(x) &= 1 + 2x + 2\frac{x^2}{2!} + 0\frac{x^3}{3!} - 4\frac{x^4}{4!} - 8\frac{x^5}{5!} - 8\frac{x^6}{6!} + 0\frac{x^7}{7!} + 16\frac{x^8}{8!} \\
&= 1 + 2x + x^2 - \frac{x^4}{6} - \frac{x^5}{15} - \frac{x^6}{90} + \frac{x^8}{2520}.
\end{aligned}
$$

   ■

2. **Example 3.3.17** *Calculate*

$$T_{8.0}\left(\cos^2 x\right)$$

   Solution If $f(x) = \cos^2 x$ then $f'(x) = -2\cos x \sin x = -\sin 2x$. This last equality will save a lot of effort when differentiating. Leave it to the student to check that

$$T_{8.0}\left(\cos^2 x\right) = 1 - x^2 + \frac{1}{3}x^4 - \frac{2}{45}x^6 + \frac{1}{315}x^8.$$

3. **Example 3.3.18** *Let* $f(x) = \sin x$. *Calculate* $T_{5,0}f(x)$ *and use Lagrange's form of the error to prove that*

$$|\sin(0.1) - T_{5,0}(\sin(0.1))| \leq 1.38888 \times 10^{-9}.$$

Hence give $\sin 0.1$ *to 8 decimal places.*

Solution From $f^{(2)}(x) = -f(x)$ we get $f^{(n)}(x) = (-1)^{n/2}\sin x$ if n is even and $f^{(n)}(x) = (-1)^{(n-1)/2}\cos x$ if n is even. Thus

$$T_{5,0}f(x) = x - \frac{x^3}{3!} + \frac{x^5}{6!}.$$

Lagrange's form of the error states that

$$R_{n,0}f(x) = \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}.$$

In this example $\left|f^{(n+1)}(c)\right| \leq 1$ for all n and c, thus

$$|R_{n,0}f(x)| \leq \frac{|x|^{n+1}}{(n+1)!}.$$

Hence

$$|\sin(0.1) - T_{5,0}(\sin(0.1))| \leq \frac{0.1^6}{6!} = 1.38888 \times 10^{-9},$$

as claimed.

We can open this out as

$$T_{5,0}(\sin(0.1)) - 1.38888\times10^{-9} \leq \sin(0.1) \leq T_{5,0}(\sin(0.1)) + 1.38888\times10^{-9}.$$

Yet

$$\begin{aligned} T_{5,0}(\sin 0.1) &= 0.1 - \frac{1}{6}(0.1)^3 + \frac{1}{120}(0.1)^5 \\ &= 9.98334166666666667 \times 10^{-2} \end{aligned}$$

So

$$9.98334166\times10^{-2} - 1.38888\times10^{-9} \leq \sin(0.1) \leq 9.98334166\times10^{-2} + 1.38888\times10^{-9}.$$

that is,

$$0.0998334152 \leq \sin(0.1) \leq 0.098334180.$$

Looking for digits in common between the upper and lower bounds we see that to 8 decimal places $\sin 0.1$ is 0.09983341.

(In fact $\sin 0.1 = 0.0998334166468281523...$) ■

4. **Example 3.3.19** *Let $f(x) = \sin^2 x$. Calculate $T_{5,0}f(x)$ and use Lagrange's form of the error to bound $\left|\sin^2(0.1) - T_{5,0}f(0.1)\right|$.*

   Solution Repeated differentiation gives

$$\begin{aligned} f'(x) &= 2\sin x\cos x = \sin 2x & f'(0) &= 0, \\ f''(x) &= 2\cos 2x, & f''(0) &= 2. \end{aligned}$$

The next derivative gives the important relationship between derivatives, $f^{(3)}(x) = -4\sin 2x = -4f^{(1)}(x)$. Then $f^{(3)}(0) = 0$ along with

$$\begin{aligned} f^{(4)}(x) &= -4f''(x), & f^{(4)}(0) &= -8, \\ f^{(5)}(x) &= -4f'''(x) = 16f'(x) & f^{(5)}(0) &= 0. \\ f^{(6)}(x) &= 16f''(x). \end{aligned}$$

Thus

$$\begin{aligned} T_{5,0}f(x) &= 0 + 0x + 2\frac{x^2}{2} + 0\frac{x^3}{3!} - 8\frac{x^4}{4!} + 0\frac{x^5}{5!} \\ &= x^2 - \frac{1}{3}x^4. \end{aligned}$$

And Lagrange's form of the error states that, for some c between x and 0,

$$R_{5,0}f(x) = \frac{x^6}{6!}\frac{d^6}{dx^6}\sin^2 x\bigg|_{x=c} = \frac{x^6}{6!}(32\cos 2c).$$

First note that with $x = 0.1 > 0$ we have $R_{5,0}f(x) > 0$, in which case

$$\sin^2(0.1) = T_{5,0}f(0.1) + R_{5,0}f(0.1) > T_{5,0}f(0.1).$$

Yet

$$T_{5,0}f(0.1) = (0.1)^2 - \frac{1}{3}(0.1)^4 = 0.009966666666...,$$

so

$$\sin^2(0.1) > 0.0099666...$$

For any upper bound we have

$$R_{5,0}f(0.1) \leq \frac{32}{6!}(0.1)^6 \leq 4.444... \times 10^{-8}.$$

Thus

$$\begin{aligned}\sin^2(0.1) &= T_{5,0}f(0.1) + R_{5,0}f(0.1) \\ &\leq (0.1)^2 - \frac{1}{3}(0.1)^4 + \frac{32}{6!}(0.1)^6 \\ &= 0.009966711111... .\end{aligned}$$

Hence

$$0.0099666... < \sin^2(0.1) < 0.009966711111... .$$

In fact

$$\sin^2(0.1) = 0.00996671107937918444... .$$

5. An example in the notes is not as strong as it could be.

Example 3.3.20 *Use Lagrange's form for the error to show that*

$$\left|\cos^2 x - \left(1 - x^2 + \frac{1}{3}x^4\right)\right| \leq \frac{2}{45}|x|^6.$$

Hence show that

$$0.9900332\overline{8} \leq \cos^2 0.1 \leq 0.9900333\overline{7}.$$

Thus

$$\cos^2 0.1 = 0.990033$$

to 6 decimal places.

In fact $\cos^2 0.1 = 0.990033288920620816...$

Solution The observation to make is that the polynomial of degree 4 is, in fact, the Taylor polynomial of degree 5. This is because

$$1 - x^2 + \frac{1}{3}x^4 = 1 + 0x - x^2 + 0x^3 + \frac{1}{3}x^4 + 0x^5 = T_{5,0}\left(\cos^2 x\right).$$

Thus

$$\begin{aligned}\left|\cos^2 x - \left(1 - x^2 + \frac{1}{3}x^4\right)\right| &= \left|R_{5,0}\left(\cos^2 x\right)\right| = \left|\frac{f^{(6)}(c)}{6!}x^6\right| \\ &= \frac{2^5}{6!}\left|\sin 2c\right|\left|x\right|^6 \le \frac{2}{45}\left|x\right|^6 .\end{aligned}$$

The Taylor polynomial approximation to $\cos^2 x$ at $x = 0.1$ is

$$T_{5,0}\left(\cos^2 x\right)\Big|_{x=0.1} = 1 - (0.1)^2 + \frac{1}{3}(0.1)^4 = 0.9900\overline{3}.$$

The error in this approximation is

$$\frac{2}{45}\left|x\right|^6 = \frac{4}{90}(0.1)^6 = 0.0000000\overline{4}.$$

Hence

$$\cos^2 0.1 \le 0.9900\overline{3} + 0.0000000\overline{4} = 0.9900333\overline{7}$$

while

$$\cos^2 0.1 \ge 0.9900\overline{3} - 0.0000000\overline{4} = 0.9900332\overline{8}.$$

■

6. **Taylor's Theorem without an error term** would have stated that "if the first n derivatives of f exist and are continuous at a then

$$\lim_{x \to a} \frac{R_{n,a} f(x)}{(x-a)^n} = 0.\text{"} \tag{9}$$

In assuming a little more, namely that the first $\mathbf{n+1}$ derivatives of f exist (which implies continuity of $f^{(i)}$, $1 \le i \le n$) we can deduce a little more, namely how quickly $R_{n,a} f(x)/(x-a)^n$ approaches 0. See (3) or (4).

7. **Taylor's Theorem with an error implies M. V. Theorem** Putting $n = 0$ in (1), the definition of the remainder gives

$$f(x) = T_{0,a}f(x) + R_{0,a}f(x) = f(a) + R_{0,a}f(x)\,.$$

Using Lagrange's Theorem gives

$$f(x) = f(a) + f'(c)\,(x - a)$$

for some c between a and x, by (4). Rearranging,

$$\frac{f(x) - f(a)}{x - a} = f'(c)\,,$$

which is the Mean Value Theorem seen earlier. But this is **not** a proof of the Mean Value Theorem since we used the ideas of the Mean Value Theorem to prove Lagrange's form of the error, (4).

8. **Integral form of the error** An alternative form of the remainder which is sometimes useful is:

Integral Form: (Cauchy 1821) If the first $n+1$ derivatives of f exist and are continuous on an open interval containing a and x then

$$R_{n,a}f(x) = \int_a^x \frac{f^{(n+1)}(t)}{n!}\,(x-t)^n\,dt.$$

(We are jumping the gun here since we have to wait until the next chapter before we define integration!) Note that we have to assume that $f^{(n+1)}(t)$ not only exists but is continuous on (a, x). This is more than is required for either Lagrange's or Cauchy's forms of the error.

9. **A limit for** $\cos x$**.** Taylor's Theorem in the form

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^3}{3!}\sin c$$

for some c between 0 and x leads to

$$\lim_{x\to 0}\frac{\cos x - 1}{x^2} = -\frac{1}{2},$$

which we have seen in Part 1 of this course. But now we see that $-1/2$ arises as a coefficient in the Taylor series.

10. **Inequalities for** $\ln(1+x)$**.** If $f(x) = \ln(1+x)$ then

$$T_{n,0}f(x) = \sum_{r=1}^{n} \frac{(-1)^{r-1} x^r}{r}.$$

Lagrange's form for the Remainder term around $x = 0$ becomes

$$R_{n,0}f(x) = \frac{f^{(n+1)}(c)}{(n+1)} x^{n+1} = \frac{(-1)^n x^{n+1}}{(1+c)^n (n+1)},$$

for some c between x and 0. Since $x > -1$ for $\ln(1+x)$ to be defined we have $c > -1$ too in which case $1 + c > 0$.

Assume $x > 0$.

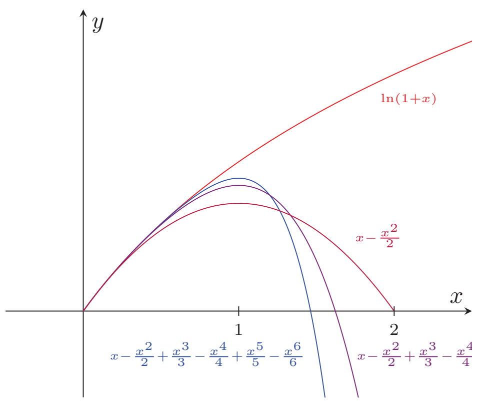
If n is even then $R_{n,0}f(x) > 0$, i.e. $\ln(1+x) > T_{n,0}f(x)$. For $n = 2, 4$ and 6 this gives

$$\ln(1+x) > x - \frac{x^2}{2}, \quad \ln(1+x) > x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}.$$

and

$$\ln(1+x) > x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6}.$$

The first of these inequalities was a problem in an earlier section deduced from the Mean Value Theorem.



If n is odd then $R_{n,0}f(x) < 0$, i.e. $\ln(1+x) < T_{n,0}f(x)$. The $n = 3$ case:

$$\ln(1+x) < x - \frac{x^2}{2} + \frac{x^3}{3}$$

was mentioned earlier in the course.

To sum up, if $x > 0$ then

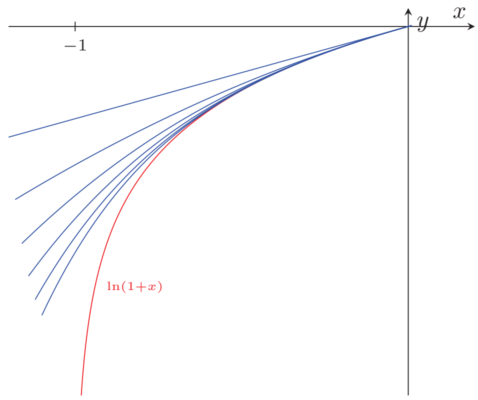
$$\ln(1+x) \begin{cases} > T_{n,0}f(x) & \text{if } n \text{ is even} \\ < T_{n,0}f(x) & \text{if } n \text{ is odd.} \end{cases}$$

Assume $-1 < x < 0$ then $(-1)^n x^{n+1} = (-x)^n x < 0$ for all n so

$$\ln(1+x) < T_{n,0}f(x)$$

for all n.

In the following diagram the $T_{n,0}f(x)$, $n = 1, ..., 6$ are plotted, increasingly better approximations to $\ln(1+x)$.



Note that for n odd we have $\ln(1+x) < T_{n,0}f(x)$ for all $x > -1$.

11. **Inequalities for** e^x. If $f(x) = e^x$ then for $x \in \mathbb{R}$ Lagrange's form of the error states that

$$R_{n,0}f(x) = \frac{e^c x^{n+1}}{(n+1)!}$$

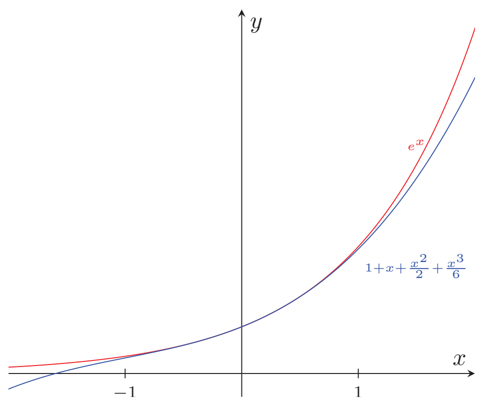
for some c between 0 and x. Whatever c, $e^c \geq 0$. Thus, when n is odd we have $x^{n+1} \geq 0$ for all x, i.e. $e^x \geq T_{n,0}f(x)$. The example when $n = 3$:

$$e^x > 1 + x + \frac{x^2}{2} + \frac{x^3}{6} \tag{10}$$

for all $x \in \mathbb{R}$ was an example left to the Student earlier in the course.

For even n we have

$$e^x \begin{cases} > T_{n,0}f(x) & \text{if } x > 0 \\ < T_{n,0}f(x) & \text{if } x < 0. \end{cases}$$



12. **Example 3.3.21** *The Taylor Series for* $\sin x$ *around* 0 *is*

$$\sum_{r=0}^{\infty} \frac{(-1)^r x^{2r+1}}{(2r+1)!}.$$

The ratio test would show that this converges for all $x \in \mathbb{R}$*, but we have to go further and show that, for each* x*,* ***it converges to*** $\sin x$*.*

Solution If $f(x) = \sin x$ then $f^{(1)}(x) = \cos x$ and $f^{(2)}(x) = -\sin x = -f(x)$. Thus, if n is even then $f^{(n)}(0)$ is a multiple of $f(0) = 0$. So the only non-zero terms have n odd, i.e. $n = 2r+1$ for $r \geq 0$. Further,

$$f^{(2r+1)}(0) = (-1)^r f^{(1)}(0) = (-1)^r.$$

The Taylor Series for $\sin x$ is

$$\sum_{r=0}^{\infty} \frac{(-1)^r x^{2r+1}}{(2r+1)!}.$$

For convergence we examine Lagrange's form of the error term,

$$R_{n,0}(\sin x) = \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1}$$

for some c between 0 and x. Yet $\left| f^{(n+1)}(c) \right|$ is either $|\sin c|$ or $|\cos c|$ and both are ≤ 1, so

$$|R_{n,0}(\sin x)| \leq \frac{|x|^{n+1}}{(n+1)!} \to 0$$

as $n \to \infty$ for *any* $x \in \mathbb{R}$. Hence, for each fixed $x \in \mathbb{R}$, we have $\lim_{n\to\infty} R_{n,0}f(x) = 0$ and so the Taylor series converges to $\sin x$, i.e.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - ... = \sum_{r=0}^{\infty} \frac{(-1)^r x^{2r+1}}{(2r+1)!}.$$

■

Note This series can be taken as the definition of sine but this would have made some of the proofs of this course more difficult. For example, to prove $d\sin x/dx = \cos x$, we would need to be able to differentiate an infinite series term by term. And since differentiation is defined by limits this is equivalent to interchanging a limit with an infinite series, a problem mentioned earlier in the notes.

13. **Example 3.3.22** *Calculate the Taylor Series for* $\sin x$ *around* $\pi/2$.

    Solution Consider

$$\left.\frac{d^n}{dx^n}\sin x\right|_{x=\frac{\pi}{2}} = \sin\left(\frac{\pi}{2} + n\frac{\pi}{2}\right) = \begin{cases} 0 & \text{if } n \text{ odd} \\ 1 & \text{if } n = 0, 4, 8, ... \\ -1 & \text{if } n = 2, 6, 10, ... \end{cases}$$

$$= \begin{cases} 0 & \text{if } n = 2r+1 \\ (-1)^r & \text{if } n = 2r. \end{cases}$$

    Hence the Taylor Series around $\pi/2$ is

$$\sum_{r=0}^{\infty} \frac{(-1)^r}{(2r)!}\left(x - \frac{\pi}{2}\right)^{2r}.$$

    The same proof as for $\sin x$ around 0 will show that this converges to $\sin x$ for all real x.

14. **Example 3.3.23** *Calculate the Taylor series for* $f(x) = e^x \cos x$ *around* 0.

Solution With $f(x) = e^x \cos x$,

$$\begin{aligned}
f^{(1)}(x) &= e^x \cos x - e^x \sin x, \qquad \text{so } f^{(1)}(0) = 1, \\
f^{(2)}(x) &= e^x \cos x - e^x \sin x - e^x \sin x - e^x \cos x \\
&= -2e^x \sin x, \qquad \text{so } f^{(2)}(0) = 0, \\
f^{(3)}(x) &= -2e^x \sin x - 2e^x \cos x, \qquad \text{so } f^{(3)}(0) = -2, \\
f^{(4)}(x) &= -2e^x \sin x - 2e^x \cos x - 2e^x \cos x + 2e^x \sin x \\
&= -4e^x \cos x = -4f(x) \quad \text{so } f^{(4)}(0) = -4,
\end{aligned}$$

The fact that $f^{(4)}(x) = -4f(x)$ makes life easy, we start repeating ourselves.

$$\begin{aligned}
f^{(5)}(x) &= -4f^{(1)}(x) \quad \text{so } f^{(5)}(0) = -4, \\
f^{(6)}(x) &= -4f^{(2)}(x) \quad \text{so } f^{(6)}(0) = 0, \\
f^{(7)}(x) &= -4f^{(3)}(x) \quad \text{so } f^{(7)}(0) = 8, \\
f^{(8)}(x) &= -4f^{(4)}(x) = 16f(x) \quad \text{so } f^{(8)}(0) = 16.
\end{aligned}$$

So the Taylor series starts as

$$\begin{aligned}
& 1 + x + \frac{0}{2!}x^2 - \frac{2}{3!}x^3 - \frac{4}{4!}x^4 - \frac{4}{5!}x^5 + \frac{8}{7!}x^7 - \ldots \qquad (11) \\
= \; & 1 + x - \frac{1}{3}x^3 - \frac{1}{6}x^4 - \frac{1}{30}x^5 + \frac{8}{630}x^7 - \ldots
\end{aligned}$$

The question must then be whether this is the same as we would obtain from multiplying together the series for e^x and $\cos x$? Try it and see...

Question *Does the series* (11) *converge to* $e^x \cos x$*?*

Solution We first need a bound on the size of $f^{(n)}(x)$. Note that it doesn't have to be a good bound, anything of the form $\left|f^{(n)}(x)\right| \leq \kappa^n e^{|x|}$ for some constant κ will suffice.

From the first list above we see that $\left|f^{(n)}(x)\right| \leq 4e^{|x|}$ for all x and $0 \leq n \leq 4$. But then $f^{(4)}(x) = -4f(x)$ which means that

$$f^{(n)}(x) = (-4)^k \, f^{(n-4k)}(x)$$

as long as $n-4k \geq 0$. We can choose k_1 such that $0 \leq n-4k_1 < 4$ which means that

$$\left|f^{(n)}(x)\right| = 4^{k_1}\left|f^{(n-4k_1)}(x)\right| \leq 4^{k_1+1}e^{|x|}.$$

Finally $0 \leq n-4k_1$ implies $k_1+1 < n$ when $n \geq 2$. Thus, for such n we have the bound

$$\left|f^{(n)}(x)\right| \leq 4^n e^{|x|}$$

for all x. Hence, for each fixed $x \in \mathbb{R}$, there exists c between 0 and x for which

$$\begin{aligned} |R_{n,0}f(x)| &= \frac{\left|f^{(n+1)}(c)\right|}{(n+1)!}|x|^{n+1} \leq \frac{4^{n+1}e^{|c|}}{(n+1)!}|x|^{n+1} \\ &\leq e^{|c|}\frac{(4|x|)^{n+1}}{(n+1)!} \to 0 \end{aligned}$$

as $n \to \infty$ by Lemma above. Hence, for each fixed $x \in \mathbb{R}$, we have $\lim_{n\to\infty} R_{n,0}f(x) = 0$ and so the series (11) converges to $e^x \cos x$ for all $x \in \mathbb{R}$.

15. (1861) **The Binomial Expansion for** $(1+x)^t = e^{t\ln(1+x)}$, for **any** exponent $t \in \mathbb{R}$, **not** just $t \in \mathbb{N}$.

    Note though, that for general t, the function $(1+x)^t$ is only defined for $x > -1$ (for only then is $\ln(1+x)$ well-defined). Since

    $$\frac{d^n (1+x)^t}{dx^n} = t(t-1)\dots(t-n+1)(1+x)^{t-n},$$

    the Taylor Series for $(1+x)^t$ is

    $$1+tx+\frac{t(t-1)}{2!}x^2+\frac{t(t-1)(t-2)}{3!}x^3+\dots = \sum_{r=0}^{\infty}\frac{t(t-1)\dots(t-r+1)}{r!}x^r.$$

    To prove that $\lim_{n\to\infty} R_{n,0}\left((1+x)^t\right) = 0$ it transpires that it is easier to use Cauchy's form of the error. I leave it to the interested student to check this, and thus find that the Taylor Series converges to $(1+x)^t$ for $-1 < x < 1$. ■

16. **Cauchy's example of 1823**

Example 3.3.24 *The Taylor series for*

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

is

$$0 + 0x + 0\frac{x^2}{2} + 0\frac{x^3}{3!} + ...$$

which converges for all $x \in \mathbb{R}$. But it's sum is $f(x)$ only when $x = 0$.

Do this by a series of Lemmas.

Lemma A

$$\lim_{x \to 0} \frac{e^{-1/x^2}}{x^n} = 0$$

for all $n \geq 1$.

Proof Recall that for $y > 0$, we have from the series defining e^y that

$$e^y = 1 + y + \frac{y^2}{2} + \frac{y^3}{3!} + ... + \frac{y^n}{n!} + ... \geq \frac{y^n}{n!},$$

throwing away all other terms, allowable since they are positive. Apply this inequality with $y = 1/x^2$ to get

$$e^{1/x^2} \geq \frac{1}{n! x^{2n}},$$

in which case

$$\left| \frac{e^{-1/x^2}}{x^n} \right| \leq \frac{n! x^{2n}}{|x|^n} = n! \, |x|^n \to 0$$

as $x \to 0$. ■

Lemma B *For $n \geq 1$, there exist polynomials $P_n(x)$ with $\deg P_n = 2(n-1)$, such that*

$$f^{(n)}(x) = \frac{P_n(x)}{x^{3n}} e^{-1/x^2},$$

for $x \neq 0$.

Proof by induction. Left to students. ■

Lemma C *For $n \geq 1$, $f^{(n)}(0) = 0$.*

Proof by induction. Starting with $n = 1$ we find that

$$f^{(1)}(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{e^{-1/x^2}}{x} = 0,$$

by Lemma A.

For the inductive step assume the result is true for $n = k$, so $f^{(k)}(0) = 0$. Consider

$$f^{(k+1)}(0) = \lim_{x\to 0} \frac{f^{(k)}(x) - f^{(k)}(0)}{x - 0} = \lim_{x\to 0} \frac{f^{(k)}(x)}{x}$$

by the inductive hypothesis. Next, by Lemma B,

$$f^{(k+1)}(0) = \lim_{x\to 0} \frac{P_k(x)}{x^{3k+1}} e^{-1/x^2}.$$

If

$$P_k(x) = \sum_{r=0}^{2(k-1)} a_r x^r$$

then

$$\lim_{x\to 0} \frac{P_k(x)}{x^{3k+1}} e^{-1/x^2} = \sum_{r=0}^{2(k-1)} a_r \lim_{x\to 0} \frac{e^{-1/x^2}}{x^{3k+1-r}} = 0$$

by Lemma A. Hence $f^{(k+1)}(0) = 0$.

Therefore, by induction, $f^{(n)}(0) = 0$ for all $n \geq 1$. ■

Thus the Taylor Series for $f(x)$ is

$$0 + 0x + 0\frac{x^2}{2} + 0\frac{x^3}{3!} + ...$$

which converges for all $x \in \mathbb{R}$. But it's sum is $f(x)$ only when $x = 0$.■

17. **Example 3.3.25** *Show that the Taylor Series for* $f(x) = e^x (\cos x + \sin x)$ *converges to* $f(x)$ *for all* $x \in \mathbb{R}$.

    Solution We have already calculated that the Taylor series of $f(x)$ starts as

$$1 + 2x + x^2 - \frac{1}{6}x^4 - \frac{1}{15}x^5 - \frac{1}{90}x^6 + \frac{1}{2520}x^8 + ..., \qquad (12)$$

and $f^{(4)}(x) = -4f(x)$. From Taylor's Theorem with *Lagrange's* form of the error we have

$$R_{n,0}f\,(x) = \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}$$

for some c between 0 and x. As for the $e^x \sin x$ example above we can show that

$$\left|f^{(n)}(c)\right| \le 4^n e^{|c|} \le 4^n e^{|x|},$$

since we need a bound not containing the unknown c. Thus

$$|R_{n,0}f(x)| \le e^{|x|}\frac{(4x)^{n+1}}{(n+1)!} \to 0$$

as $n \to \infty$, by the Lemma above. Hence (12) converges to $e^x(\cos x + \sin x)$ for all $x \in \mathbb{R}$.

18. **Taylor series of** $\ln(1+x)$. In this course we have defined the natural logarithm as the inverse of e^x. Thus we can calculate the Taylor series of $\ln(1+x)$. First published by Mercator in 1668, the series is

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \dots .$$

The ratio test shows the series converges for $|x| < 1$, while the Alternating Series Test shows that it converges when $x = 1$. But again we have to show that it converges to $\ln(1+x)$.

Writing $f(x) = \ln(1+x)$ then

$$f^{(j)}(x) = \frac{(-1)^{j+1}(j-1)!}{(1+x)^j}$$

for all $j \ge 1$. The *integral form* of the error states

$$R_{n,0}f(x) = \int_0^x \frac{f^{(n+1)}(t)}{n!}(x-t)^n\,dt = (-1)^{n+2}\int_0^x \frac{(x-t)^n}{(1+t)^{n+1}}dt.$$

The most interesting case (because it is the most difficult) is $x = 1$ when we get the integral

$$I_n = \int_0^1 \left(\frac{1-t}{1+t}\right)^n \frac{dt}{(1+t)}.$$

Substitute $w = (1-t) / (1+t)$ to transform into

$$I_n = \int_0^1 \frac{w^n}{1+w} dw \leq \int_0^1 w^n dw = \frac{1}{n+1} \to 0.$$

as $n \to \infty$. Thus $\lim_{n\to\infty} R_{n,0} (\ln (1+x))|_{x=1} = 0$. This justifies

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + ...$$

19. **Example 3.3.26** *Find the Taylor Series for* $\sin^2 x$ *around* $a = 0$ *and show that the series converges to* $\sin^2 x$ *for all* $x \in \mathbb{R}$.

Solution When looking at the Taylor polynomial for $f(x) = \sin^2 x$ we already saw

$$\begin{aligned} f^{(1)}(x) &= 2 \sin x \cos x = \sin 2x, \\ f^{(2)}(x) &= 2 \cos 2x, \\ f^{(3)}(x) &= -4 \sin 2x = -4 f^{(1)}(x). \end{aligned}$$

From this it is easy to deduce

$$f^{(r)}(x) = \begin{cases} (-1)^{t-1} 2^{2t-2} \sin 2x & \text{if } r = 2t-1 \text{ is odd} \\ (-1)^{t-1} 2^{2t-1} \cos 2x & \text{if } r = 2t \text{ is even.} \end{cases}$$

Thus

$$f^{(r)}(0) = \begin{cases} 0 & \text{if } r \text{ is odd} \\ (-1)^{t-1} 2^{r-1} & \text{if } r = 2t \geq 2. \end{cases}$$

Therefore the Taylor series is

$$0 + 0x + \sum_{\substack{r=2 \\ r \text{ even} \\ r=2t}}^{\infty} \frac{(-1)^{t-1} 2^{r-1}}{r!} x^r = \sum_{t=1}^{\infty} \frac{(-1)^{t-1} 2^{2t-1}}{(2t)!} x^{2t}.$$

The first few terms are

$$x^2 - \frac{1}{3}x^4 + \frac{2}{45}x^6 - \frac{1}{315}x^8 + ...$$

To show that the series converges to $\sin^2 x$ for all $x \in \mathbb{R}$ we need show that

$$\lim_{n\to\infty} R_{n,0} \left(\sin^2 x\right) = 0$$

for all $x \in \mathbb{R}$. To do this we use Lagrange's form of the error so, for any $x \in \mathbb{R}$ we have

$$R_{n,0}\left(\sin^2 x\right) = \frac{f^{(n+1)}(c)}{(n+1)!}x^n \tag{13}$$

for some c between 0 and x. From above

$$\left|f^{(r)}(x)\right| \leq \begin{cases} \left|2^{2t-2}\right| = 2^{r-1} & \text{if } r = 2t-1 \text{ is odd} \\ \left|2^{2t-1}\right| = 2^{r-1} & \text{if } r = 2t \text{ is even.} \end{cases}$$

Thus

$$\left|f^{(r)}(x)\right| \leq 2^{r-1}$$

for $r \geq 1$. Hence (13) becomes

$$\left|R_{n,0}\left(\sin^2 x\right)\right| = \frac{\left|f^{(n+1)}(c)\right|}{(n+1)!}\left|x\right|^n \leq \frac{2^n}{(n+1)!}\left|x\right|^n = \frac{(2\left|x\right|)^n}{(n+1)!} \to 0$$

as $n \to \infty$ by Lemma above. Thus $R_{n,0}\left(\sin^2 x\right) \to 0$ as $n \to \infty$ and the series converges to $\sin^2 x$ for all $x \in \mathbb{R}$. ■